MATH 412: PARTIAL DIFFERENTIAL EQUATIONS.

Course outline.

Introduction of basic concept. Boundary and initial value problems. Theory and solution of first and second order linear equations. Wave, heat and Laplace equation in cartesian and polar coordinates. Classification of partial differential equation (PDE), characteristics and cannonical forms. Cauchy problems. Elliptic equations. Laplace and poisson equations. Solution in Cylindrical polar and spherical polar coordinates. Hyperbolic and parabolic equations. Wave and diffusions. Green's function, harmonic function properties.

Reference Books.

1. Dass. H.K. Advanced Eng. Mathematics / Eng.
2. K.A Stroud: Further engineering mathematics.
3. Spiegel, M.R Advanced mathematics for scientist and engineers.
4. Differential equation (Schaum's series).
5. Kreyzig, E. Advanced engineering mathematics.

Definition

A partial differential equation (PDE) is an equation containing the partial derivative(s) of an unknown function with respect to the independent variables.

E.g

1. $\partial u/\partial x = 3\,\partial u/\partial y + y$
2. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \partial^2 z/\partial t^2$
3. $\partial^3 z/\partial x^3 + 3\,\partial^2 z/\partial y^2 + \left(\partial z/\partial y\right)^2 + \left(\partial z/\partial x\right)^2 = xy$
4. $\partial^4 z/\partial y^4 + x\,\partial^3 z/\partial x^3 = \partial^4 z/\partial x^4$

classification of partial differential equation.

1. Classical and Non-classical PDE.

The classical differential equations are equation used in practical applications. E.g: the heat - equation: ① $\partial u/\partial t = 2\,\partial^2 u/\partial x^2$, $\partial u/\partial z = \alpha(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2)$

② Wave equation: $\partial^2 y/\partial t^2 = \alpha\,\partial^2 y/\partial x^2$,

③ Laplace equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

$\partial u/\partial t = \alpha(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial t^2)$: $\partial u/\partial t = \alpha\nabla^2 u$, $u \equiv u(x,u,t)$. If $\partial u/\partial t = 0 \Rightarrow \alpha\nabla^2 u = 0$ or $\nabla^2 u = 0$. $\partial^2 u/\partial t^2 = \partial\nabla^2 u$, $\partial u/\partial t = 0 \Rightarrow \nabla^2 u = 0$

- Laplace equation:
- Poission equation

- Mass conservation
- Euler's Momentum equation.

Formation of partial Differential equation

A partial differential equation can be obtained from a given function primitive by eliminating the constant or the unknown function.

Example:

Form PDE from the following:

① $z = ax + by$

Diff w.r.t x and $\Rightarrow \partial z/\partial x = a$, $\frac{\partial z}{\partial y} = b$

$\Rightarrow z = x\,\partial z/\partial x + y\,\partial z/\partial y$

②. $\phi(x^2+y^2, x+y) = 0$

Let $u = x^2+y^2$ and $v = x+y \Rightarrow \phi(u,v) = 0$

Differentiate w.r.t x and y

$\partial\phi/\partial u \cdot \partial u/\partial x + \partial\phi/\partial v \cdot \partial v/\partial x = \partial\phi/\partial u (2x) + \partial\phi/\partial v (1) = 0$

$\Rightarrow 2x\,\partial\phi/\partial u + \partial\phi/\partial v = 0$ —— ①

w.r.t y

$y\ \partial\phi/\partial u (2y) + \partial\phi/\partial v (1) = \cancel{0}\ 0$

$= 2y\,\partial\phi/\partial u + \partial\phi/\partial v = 0$ —— ②

$$\begin{pmatrix} 2x & 1 \\ 2y & 1 \end{pmatrix}\begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2x & 1 \\ 2y & 1 \end{vmatrix} \begin{vmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{vmatrix} = 0$$

③ Form the differential equation from $\phi(x^2+y^2, z-xy)=0$

Let $u = x^2+y^2$, $v = z-xy$. $\phi(u,v)=0$

diff. w.r.t x

$$\Rightarrow \partial\phi/\partial u \cdot \partial u/\partial x + \partial\phi/\partial v \cdot \partial v/\partial x = 0$$

$$\frac{\partial\phi}{\partial u}(2x) + \frac{\partial\phi}{\partial v}(z_x - y) = 0$$

$$2x\,\partial\phi/\partial u + (z_x - y)\,\partial\phi/\partial v = 0 \quad \text{—— ①}$$

similarly

diff. w.r.t y $\Rightarrow \partial\phi/\partial u \cdot \partial u/\partial y + \partial\phi/\partial v \cdot \partial v/\partial y = 0$

$$\Rightarrow \partial\phi/\partial u\,(2y) + \partial\phi/\partial v\,(z_y - x) = 0$$

$$\Rightarrow 2y\,\partial\phi/\partial u + (z_y - x)\,\partial\phi/\partial v = 0 \quad \text{—— ②}$$

$$\begin{pmatrix} 2x & z_x - y \\ 2y & z_y - x \end{pmatrix} \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix} = 0 \quad \therefore \begin{vmatrix} 2x & z_x - y \\ 2y & z_y - x \end{vmatrix} = 0$$

$$= 2x(z_y - x) - 2y(z_x - y) = 0$$

$$\Rightarrow 2x z_y - 2x^2 - 2y z_x + 2y^2 = 0$$

$\therefore x z_y - y z_x = x^2 - y^2$ is the required PDE

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = x^2 - y^2$$

④ Form PDE from the following fn.

$\phi(xy + z^2, y + z) = 0$

Let $u = xy + z^2$, $v = y + z \Rightarrow \phi(u, v) = 0$

$\partial\phi/\partial u \cdot \partial u/\partial x + \partial\phi/\partial v \cdot \partial v/\partial x = 0$

$\partial\phi/\partial u (1 \cdot y + 2z \cdot z_x) + \partial\phi/\partial v (z_x) = 0$

$(y + 2z \cdot z_x)\, \partial\phi/\partial u + z_x\, \partial\phi/\partial v = 0$ —— ①

$\partial\phi/\partial u \cdot \partial u/\partial y + \partial\phi/\partial v \cdot \partial v/\partial y = 0$

$\Rightarrow \partial\phi/\partial u (1 \cdot x + 2 \cdot z \cdot z_y) + (1 + z_y)\, \partial\phi/\partial v = 0$

$\Rightarrow (x + 2z \cdot z_y)\, \partial\phi/\partial u + (1 + z_y)\, \partial\phi/\partial v = 0$

$$\begin{pmatrix} y + 2z \cdot z_x & z_x \\ x + 2z \cdot z_y & 1 + z_y \end{pmatrix} \begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} \Rightarrow \begin{vmatrix} y + 2z \cdot z_x & z_x \\ x + 2z \cdot z_y & 1 + z_y \end{vmatrix} = 0$$

$(y + 2z \cdot z_x)(1 + z_y) - z_x(x + 2z \cdot z_y) = 0$

$y + y z_y + 2z z_x + 2z z_x z_y - x z_x z_y - x z_x$
$- 2z z_x z_y = 0$

$y + y z_y + 2z z_x - x z_x = 0$

$(2z - x) z_x + y(1 + z_y) = 0$

Exercise II :

Form PDE from the following :

i. $z = (a + x^2)(b + y^2)$

ii. $\phi(z/x^2, x - y) = 0$

iii. $\phi(x + y + z, x^2 + y^2 + z^2)$

iv. $z = f(x - at) + g(x + at)$

v. $z = (x + y)\, f(x^2 + y^2)$

Solution of partial differential equation

A function is said to be solution to a given partial differential equation if the equation is satisfied whenever the function substituted in to the equation (PDE).

Methods of solving partial differential equations

These methods include:

1. Direct method
2. Method of separation of parameters
3. Laplace transform method
4. Fourier series method.
5. Fourier transform and fourier integral's method

Direct method.

In this method, the given partial differential is solved by successive integration.

Example

1. Solve the equation:

$\partial^2 u/\partial x \partial y = 4e^{y}\cos 2x$, given that at $y=0$, $\partial u/\partial x = \cos 2x$ at $x=\pi$, $u=y^2$

Soln

$\partial^2 u/\partial x\partial y = 4e^{y}\cos 2x \Rightarrow \partial/\partial y\left(\partial u/\partial x\right) = 4e^{y}\cos 2x$

Integrate w.r.t y

$\partial u/\partial x = 4e^y \cos 2x + c(x)$

since at $y=0$, $\partial u/\partial x = \cos 2x \Rightarrow \cos 2x = 4\cos 2x + c(x)$

$\therefore c(x) = \cos 2x - 4\cos 2x = -3\cos 2x$

① $\partial u/\partial x = 4e^y \cos 2x - 3\cos 2x$

integrate w.r.t x

$u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + G(y)$

$\therefore$ at $x=\pi$, $u=y^2 \Rightarrow y^2 = 2e^y \sin 2\pi - 3/2 \sin 2\pi + G(y)$

$G(y) = y^2$

$\therefore u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + y^2$

2. Separation of parameters methods.

In this method the unknown function is assumed to be a product of functions each of the independent variable; and the equation (given) is reduced to "n" number of ordinary differential equations and is then solved accordingly.

Example

1. solve the equation $\partial u/\partial t = 2\,\partial^2 u/\partial x^2$, $u(0,t)=0$, $u(2,t)=0$, $u(x,0) = 3\sin 2\pi x - 5\sin 4\pi x$, $0 \le x \le 2$, $t \ge 0$

soln

$\partial u/\partial t = 2\partial^2 u/\partial x^2$, Let $u(x,t) = X(x) \cdot T(t) = XT$

$\therefore \partial u/\partial t = XT'$,

$\partial^2 u/\partial x^2 = X''T$ $\qquad T' + 2\alpha^2 T = 0$

$\Rightarrow XT' = 2X''T$

$$\frac{T'}{2T} = \frac{X''}{X} = -\alpha^2 \Rightarrow \frac{T'}{2T} = -\alpha^2 \quad \text{---(1)}$$

$$X'' + \alpha^2 X = 0 \quad \text{---(2)}$$

From (1) $\dfrac{T'}{T} = -2\alpha^2 \Rightarrow \ln T = -2\alpha^2 t + A$

$\Rightarrow T = e^{-2\alpha^2 t + A} = e^{A} \cdot e^{-2\alpha^2 t} = ke^{-2\alpha^2 t}$

$\therefore T(t) = ke^{-2\alpha^2 t}$

Similarly, from (2) $X'' + \alpha^2 X = 0$

The auxilliary equation is $m^2 + \alpha^2 = 0$ $\therefore m = \pm \alpha i$

$X(x) = A_1 \cos \alpha x + A_2 \sin \alpha x$

$\therefore u(x,t) = XT = ke^{-2\alpha^2 t}(A_1 \cos \alpha x + A_2 \sin \alpha x)$

$u(x,t) = (A \cos \alpha x + B \sin \alpha x) e^{-2\alpha^2 t}$

$u(0,t) = 0 \Rightarrow (A \cdot \cos \alpha \cdot 0 + B \sin \alpha \cdot 0) e^{-2\alpha^2 t} = 0$

$\Rightarrow Ae^{-2\alpha^2 t} = 0 \Rightarrow A = 0$

then $u(2,t) = 0 \Rightarrow B \sin 2\alpha \, e^{-2\alpha^2 t} = 0 \Rightarrow \sin 2\alpha = 0$

$\Rightarrow 2\alpha = n\pi$, $n = 0, 1, 2, \ldots \Rightarrow \alpha = n\pi/2$

$u_n(x,t) = B_n \sin \frac{n\pi x}{2} \, e^{-\frac{2n^2\pi^2}{4} t}$

$u_n(x,t) = B_n \sin \frac{n\pi x}{2} \, e^{-\frac{n^2 \pi^2}{2} t}$

By the principle of superposition. i.e If $y_1, y_2, \ldots, y_n$ are the solution of the equation, then the linear combination is also a solution.

i.e $y(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

Thus, $u(x, t) = c_1 u_1 + c_2 u_2$

$\therefore u(x, t) = B_1 \sin \frac{n_1 \pi x}{2} e^{-\frac{n_1^2 \pi^2 t}{2}} + B_2 \sin \frac{n_2 \pi x}{2} e^{-\frac{n_2^2 \pi^2 t}{2}}$

$\therefore u(x,t) = B_1 \sin \frac{n_1 \pi x}{2} + B_2 \sin \frac{n_2 \pi x}{2} = 3 \sin 2\pi x - 5 \sin 4\pi x$

$B_1 = 3,\ B_2 = -5,\ n_1/2 = 2 \Rightarrow n_1 = 4,\ n_2/2 = 4 \Rightarrow n_2 = 8$

$\therefore u(x, t) = 3 \sin 2\pi x\, e^{-8\pi^2 t} - 5 \sin 4\pi x\, e^{-32\pi^2 t}$

(2) Show the PDE: $\partial^2 y/\partial t^2 = 4 \partial^2 y/\partial x^2$, $y(0,t)$, $y(0,t) = y(\pi, t) = 0$, $\cancel{y}\ \partial y/\partial t (x,0) = 0$, $y(x, 0) = 2 \sin 5x$. $0 \le x \le \pi$, $t \ge 0$.

$y(x,t) = XT$

$XT'' = 4X''T \Rightarrow \frac{T''}{4T} = \frac{X''}{X} = -\alpha^2$

$T'' + 4\alpha^2 T = 0$ ——— (1)

$X'' + \alpha^2 X = 0$ ——— (2)

from (1), the a.e is $m^2 + 4\alpha^2 = 0$, $\therefore m = \pm 2\alpha i$

$T(t) = A_1 \cos 2\alpha t + A_2 \sin 2\alpha t$

From (2) $X(x) = B_1 \cos \alpha x + B_2 \sin \alpha x$

$y(x, t) \underset{\uparrow}{\overset{= XT}{=}} (A_1 \cos 2\alpha t + A_2 \sin 2\alpha t)(B_1 \cos \alpha x + B_2 \sin \alpha x)$

$y(0,t)=0 \Rightarrow (A_1 \cos 2\alpha t + A_2 \sin 2\alpha t)(B_1 \cdot \cos\alpha \cdot 0 + B_2 \sin \alpha \cdot 0)$

$\therefore (A_1 \cos 2\alpha t + A_2 \sin 2\alpha t) B_1 = 0 \Rightarrow B_1 = 0$

$y(x,t) = B_2 \sin\alpha x (A_1 \cos 2\alpha t + A_2 \sin 2\alpha t)$

or $y(x,t) = \sin\alpha x (A \cos 2\alpha t + B \sin 2\alpha t)$

$dy/dt = (-2\alpha A \sin 2\alpha t + 2\alpha B \cos 2\alpha t) \sin\alpha x$

$\therefore \partial y/\partial t (x,0) = 0 \Rightarrow (-2\alpha A \sin 2\alpha \cdot 0 + 2\alpha B \cos 2\alpha \cdot 0) \sin\alpha x$

$\Rightarrow 2\alpha B \sin\alpha x = 0 \Rightarrow B = 0$

$y(x,t) = A \sin\alpha x \cos 2\alpha t$

Similarly

$y(\pi, t) = 0 \Rightarrow A \sin\alpha\pi \cos\alpha t = 0$

$\therefore \sin\alpha\pi = 0 \Rightarrow \alpha = n, \quad n = 0, 1, 2, \ldots$

$y_n(x,t) = A_n \sin nx \cos 2nt, \quad n = 0, 1, 2, \ldots$

$y(x,0) = 2\sin 5x \Rightarrow A_n \sin nx = 2 \sin 5x$

$\Rightarrow A_n = 2, \quad n = 5$

$y(x,t) = 2\sin 5x \cos 10t$

④ Solve $\partial u/\partial x = 4 \partial u/\partial y$, $u(0,y) = 8e^{-3y}$

Soln

Let $u(x,y) = X(x)Y(y)$

Substituting into the given equation we have

$X'Y = 4XY' \Rightarrow \dfrac{X'}{4X} = \dfrac{Y'}{Y}$

$\Rightarrow \dfrac{X'}{X} = 4\alpha, \quad \dfrac{Y'}{Y} = \alpha, \Rightarrow \ln X = 4\alpha x + A$

$\therefore X = e^{4\alpha x + A} = K_1 e^{4\alpha x}$ and

$\ln Y = \alpha y + B \Rightarrow Y = e^{\alpha y + B} = K_2 e^{\alpha y}$

$u(x,y) = XY = K_1 e^{4\alpha x} \cdot K_2 e^{\alpha y} = K e^{\alpha(4x+y)}$

$u = K e^{\alpha(4x+y)}$, $\therefore u(0,y) = 8e^{-3y}$

$\Rightarrow K e^{\alpha y} = 8e^{-3y} \Rightarrow K = 8, \alpha = -3$

$u(x,y) = 8e^{-3(4x+y)}$

Fourier Series Method

If $f(x)$ is defined in the interval $(-L, L)$ and satisfies the Dirichlet conditions, then $f(x)$ can be expanded in a series of the fourier

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos nx \, dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin nx \, dx$$

If $f(x)$ is defined in the half interval $[0, L]$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$; $a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{2} dx$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \Rightarrow b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$f(x) = a_0/2 + a_1 \cos n\pi x + a_2 \cos n\pi x + \cdots$ is called cosine (fourier) series.

example.

solve the partial differential equation

$\partial u/\partial t = 2\partial^2 u/\partial x^2$, $u(0,t) = u(3,t) = 0$, $u(x,0) = 5x$

$0 \le x \le 3$, $t \ge 0$

soln

Let $u(x,t) = XT \Rightarrow XT' = 2X''T$

$$\frac{T'}{2T} = \frac{X''}{X} = -\alpha^2 \Rightarrow T(t) = A_1 e^{-2\alpha^2 t}$$

$$X'' + \alpha^2 X = 0 \Rightarrow X(x) = (B_1 \cos\alpha x + B_2 \sin\alpha x)$$

$$u(x,t) = A_1 e^{-2\alpha^2 t}(B_1\cos\alpha x + B_2 \sin\alpha x) = e^{-2\alpha^2 t}(A\cos\alpha x + B\sin\alpha x)$$

$$u(x,t) = e^{-2\alpha^2 t}(A\cos\alpha x + B\sin\alpha x)$$

$u(0,t) = 0 \Rightarrow A = 0$. $u(x,t) = Be^{-2\alpha^2 t}\sin\alpha x$

$u(3,t) = 0 \Rightarrow Be^{-2\alpha^2 t}\sin 3\alpha = 0 \Rightarrow \sin 3\alpha = 0$ or

$3\alpha = n\pi \Rightarrow \alpha = n\pi/3$, $n = 0, 1, 2, \ldots$

~~$\therefore u(x,t) = \sum B_n$~~

$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{2n^2\pi^2}{9}t} \sin\frac{n\pi}{3}x$$

$$\therefore u(x,0) = 5x \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{3}x$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

$$\Rightarrow B_n = \frac{2}{3}\int_0^3 5x\sin\frac{n\pi}{3}x\,dx = \frac{10}{3}\int_0^3 x\sin\frac{n\pi}{3}x\,dx.$$

$$= 10/3\left[x \cdot \frac{-\cos\frac{n\pi x}{3}}{n\pi/3} + \int_0^3 \frac{3}{n\pi}\cos\frac{n\pi x}{3}\,dx\right]$$

$$= 10/3\left[\frac{-3x}{n\pi}\cos\frac{n\pi x}{3}\Big|_0^3 + \frac{3}{n\pi}\left(\frac{\sin n\pi/3\, x}{n\pi/3}\Big|_0^3\right)\right]$$

$$= 10/3\left[\frac{-9}{n\pi}\cos n\pi + \frac{9}{n^2\pi^2}\sin n\pi\right]$$

$$\therefore\ B_n = -30/n\pi \cos n\pi$$

$$u(x,t) \therefore f(x) = \sum_{n=1}^{\infty} -\frac{30}{n\pi}\cos n\pi\, e^{-\frac{2n^2\pi^2}{9}t}\sin\frac{n\pi}{3}x$$

$$u(x,t) = \frac{-30}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}e^{-\frac{2n^2\pi^2}{9}t}\sin\frac{n\pi x}{3}$$

$$u(x,t) = \frac{30}{\pi}\left(e^{-\frac{2\pi^2}{9}t}\sin\frac{\pi}{3}x - e^{-8/9\pi^2}\sin 2\pi/3\,x + \cdots\right.$$

Example

solve the Laplace equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

$u(0,y) = 0$, $u(x,0) = 0$, $u(5,y) = 0$, $u(x,2) = 10x$,

$0 \le x \le 5$, $0 \le y \le 2$.

Soln

Let $u(x,y) = XY \Rightarrow X''Y + XY'' = 0$

dividing through by XY

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2$$

$$X'' + \alpha^2 X = 0, \quad Y'' - \alpha^2 Y = 0$$

$\Rightarrow X(x) = A_1 \cos \alpha x + A_2 \sin \alpha x, \; Y = B_1 \cosh \alpha y + B_2 \sinh \alpha y$

$\therefore u(x,y) = (A_1 \cos \alpha x + A_2 \sin \alpha x)(B_1 \cosh \alpha y + B_2 \sinh \alpha y)$

$u(0,y) = 0 \Rightarrow A_1(B_1 \cosh \alpha y + B_2 \sinh \alpha y) = 0, \; \therefore A_1 = 0$

$u(x,y) = A_2 \sin \alpha x (B_1 \cosh \alpha y + B_2 \sinh \alpha y)$

$u(x,y) = \sin \alpha x (A \cosh \alpha y + B \sinh \alpha y)$

$\therefore u(x,0) = 0 \Rightarrow \sin \alpha x (A) = 0 \; \therefore A = 0$

$u(x,y) = B \sin \alpha x \sinh \alpha y$

$u(5,y) = 0 \Rightarrow B \sin 5\alpha \sinh \alpha y = 0$

$\Rightarrow \sin 5\alpha = 0 \Rightarrow 5\alpha = n\pi \Rightarrow \alpha = n\pi/5, \; n = 0, 1, 2 \ldots$

$u(x,2) = 10x \Rightarrow 10x = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{5} x \sinh \frac{2n\pi}{5}$

Let $B_n \sinh \frac{2n\pi}{5} = C_n$

$\therefore 10x = \sum_{n=0}^{\infty} C_n \sin \frac{n\pi x}{5}$

$\Rightarrow C_n = \frac{2}{5} \int_0^5 10x \sin \frac{n\pi}{5} x \, dx = 4 \int_0^5 x \sin \frac{n\pi x}{5} dx$

$= 4 \left[\frac{-5x}{n\pi} \cos \frac{n\pi x}{5} \Big|_0^5 + \frac{5}{n\pi} \int_0^5 \cos \frac{n\pi}{5} x \, dx \right.$

$\Rightarrow C_n = -\frac{100}{n\pi} \cos n\pi$

$B_n \sinh \frac{2n\pi}{5} = -\frac{100}{n\pi} \cos n\pi \Rightarrow B_n = \frac{-100 \cos n\pi}{n\pi \sinh \frac{2n\pi}{5}}$

$u(x,y) = \sum \frac{-100 \cos n\pi}{n\pi \sinh \frac{2n\pi}{5}} \sin \frac{n\pi x}{5} \sinh \frac{n\pi y}{5}$

Exercise.

1. solve the equation $U_{xx} + U_{yy} = 0$ given that $U(0,y) = U(x,0) = 0$, $U(x,2) = 0$, $U(3,y) = 2y+1$, $0 \le x \le 3$, $0 \le y \le 2$.

2. $4\, \partial u/\partial x + \partial u/\partial y = 3u$, If $u = 3e^{-y} - e^{-5y}$ or $u(0,y) = 3e^{-y} - e^{-5y}$ where $x=0$

3. $\partial^2 u/\partial x^2 = 24x^2(t-2)$, given that at $x=0$, $u = e^{2t}$ and $\partial u/\partial x = 4t$

4. $\partial^2 y/\partial t^2 = 4\, \partial^2 y/\partial x^2$, If $y(x,0) = 3x$, $y(0,t) = 0$, $y(\pi,t) = 0$; and $\partial y/\partial t(x,0) = 0$.

3. Laplace transform method.

In this method the given PDE is reduced to an ordinary differential equation (in terms of the other variables. The resulting O.D.E is solved accordingly. Hence the solution of the PDE is obtained by determing the inverse Laplace transform of the solution of the O.D.E.

Examples.

1- Use the Laplace transform method to solve the PDE: $\partial u/\partial t = \partial u/\partial x - 2u$, $u(0,t) = 10e^{-3t}$, $u(x,0) = 10e^{-x}$.

$$\mathcal{L}\,\partial u/\partial x = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\,dt = \frac{d}{dx}\int_0^\infty u e^{-st}\,dt = \frac{d}{dx}\hat{u} - u(0) \qquad ye^{\int p\,dx} = \int Q(x)\,e^{\int p\,dx}$$

solution.

Taking the Laplace transform of the ~~equation~~ given equation, w.r.t t, we have

$$\mathcal{L}\,\partial u/\partial t = \mathcal{L}\,\partial u/\partial x - 2\mathcal{L}\,u$$

$$s\hat{u}(s) - u(x,0) = \frac{d\hat{u}}{dx} - 2\hat{u}$$

$$s\hat{u}(x,s) - 10e^{-x} = \frac{d\hat{u}}{dx} - 2\hat{u}$$

$$\Rightarrow \frac{d\hat{u}}{dx} - (2+s)\hat{u} = -10e^{-x}$$

$$\hat{u}e^{-\int(s+2)dx} = \int -10e^{-x}\cdot e^{-(s+2)x}\,dx + c(t)$$

$$\hat{u}e^{-(2+s)x} = -10\int e^{-(s+3)x}\,dx + c(t)$$

$$\hat{u}e^{-(2+s)x} = -10\frac{e^{-(s+3)x}}{-(s+3)} + c$$

$\cancel{\hat{u}e^{-(2+s)x}}$

$$\hat{u}e^{-(2+s)x} = \frac{10}{s+3}e^{-x} + c$$

$$\hat{u}(x,s) = \frac{10}{s+3}e^{-x} + ce^{(s+2)x}$$

$$\mathcal{L}\,u(0,t) = \mathcal{L}\,10e^{-3t} = 10\,\mathcal{L}\,e^{-3t} = 10\cdot\frac{1}{s+3} = 10/s+3$$

$$\Rightarrow \hat{u}(0,s) = 10/s+3$$

$$\Rightarrow \frac{10}{s+3} = \frac{10}{s+3} + c \Rightarrow c = 0 \Rightarrow \hat{u}(x,s) = \frac{10}{s+3}e^{-x}$$

$$\therefore u(x,t) = \mathcal{L}^{-1}\hat{u}(x,s) = \mathcal{L}^{-1}\frac{10}{s+3}e^{-x} = e^{-x}\cdot\mathcal{L}^{-1}\frac{10}{s+3}$$

$\Rightarrow u(x,t) = e^{-x} \cdot 10\, \mathcal{L}^{-1} \frac{1}{s+3} = 10 e^{-x} \cdot e^{-3t}$

$u(x,t) = 10^{-(x+3t)}$

② solve the equation $\partial u/\partial t = 4\partial^2 u/\partial x^2$, $u(0,t)=0$

$u(3,t)=0$, $u(x,0) = 5\sin 2\pi x$. $t \geq 0$, $0 \leq x \leq 3$

sol^n

Taking the Laplace transform with respect to t give

$\mathcal{L}\, \partial u/\partial t = 4 \mathcal{L}\, \partial^2 u/\partial x^2 \Rightarrow s\hat{u}(x,s) - u(x,0) = 4\frac{d^2\hat{u}}{dx^2}$

$\Rightarrow 4\frac{d^2\hat{u}}{dx^2} - s\hat{u} = -5\sin 2\pi x$; $\hat{u}(0,s)=0$, $\hat{u}(3,s)=0$

$\mathcal{L}\, u(0,t)=0 \Rightarrow \hat{u}(0,s)=0$, $\mathcal{L}\, u(3,t)=0 \Rightarrow \hat{u}(3,s)=0$

$\hat{u} = \int_0^\infty e^{-st} u\,dt$, $s>0$, $t \geq 0$

The a.e of the ODE is $4m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}/2$

$\hat{u}(x,s) = A_1 e^{x\sqrt{s}/2} + A_2 e^{-x\sqrt{s}/2}$

$up = 1/4D^2 - s\,(-5\sin 2\pi x)$

$up = \frac{1}{4(-2\pi^2)-s}(-5\sin 2\pi x) = \frac{5\sin 2\pi x}{s+8\pi^2}$

$\hat{u}(x,s) = A_1 e^{x\sqrt{s}/2} + A_2 e^{-x\sqrt{s}/2} + \frac{5\sin 2\pi x}{s+8\pi^2}$

$\hat{u}(0,s)=0 \Rightarrow A_1 + A_2 = 0 \Rightarrow A_2 = -A_1$

$\hat{u}(x,s) = A_1(e^{x\sqrt{s}/2} - e^{-x\sqrt{s}/2}) + \frac{5\sin 2\pi x}{s+8\pi^2}$

$$\hat{u}(3,s)=0 \Rightarrow A_1\left(e^{3\sqrt{s}/2}-e^{-3\sqrt{s}/2}\right)+\frac{5\sin 6\pi}{s+8\pi^2}=0$$

$$\Rightarrow A_1\left(e^{3\sqrt{s}/2}-e^{-3\sqrt{s}/2}\right)=0 \Rightarrow A_1\left(e^{6\sqrt{s}/2}-1\right)=0 \Rightarrow A_1=$$

$$\hat{u}(x,s)=\frac{5\sin 2\pi x}{s+8\pi^2}$$

$$\therefore \hat{u}(x,t)=\mathcal{L}^{-1}\hat{u}(x,s)=\mathcal{L}^{-1}\frac{5\sin 2\pi x}{s+8\pi^2}$$

$$=5\sin 2\pi x\,\mathcal{L}^{-1}\frac{1}{s+8\pi^2}$$

$$=5\sin 2\pi x\, e^{-8\pi^2 t}$$

Hence

$$u(x,t)=5\sin 2\pi x\, e^{-8\pi^2 t}$$

③ $\partial u/\partial t = 2\,\partial^2 u/\partial x^2$, $\partial u/\partial x(0,t)=\partial u/\partial x(3,t)=0$

$u(x,0)=3\cos 2\pi x$

Sol^n

$$2\frac{d^2\hat{u}}{dx^2}-s\hat{u}=-3\cos 2\pi x$$

$$\mathcal{L}\,\partial u/\partial x(0,t)=0 \Rightarrow \mathcal{L}\,\partial/\partial x\, u(0,t)=\frac{d}{dx}\hat{u}(0,s)=0$$

$$\mathcal{L}\,\frac{\partial u}{\partial x}(3,t)=0 \Rightarrow \frac{d}{dx}\mathcal{L}\,u(3,t)=\frac{d}{dx}\hat{u}(3,s)=0$$

$$\Rightarrow \frac{d\hat{u}}{dx}(0,s)=\frac{d\hat{u}}{dx}(3,s)=0$$

$2m^2 - s = 0 \quad \therefore m = \pm\sqrt{s/2}$

$\hat{u}(x,s) = A_1 e^{\sqrt{s/2}\,x} + A_2 e^{-\sqrt{s/2}} + 3\cos 2\pi x$

$\frac{d\hat{u}}{dx}(0,s) = 0 \Rightarrow \sqrt{s/2}\,(A1.1 - A_2) - 6\sin 2\pi.0 = 0$

$\therefore \sqrt{s/2}\,(A_1 - A_2) = 0 \Rightarrow A_2 = A_1$

$\hat{u}(x,s) = A_1\left(e^{\sqrt{s/2}\,x} + e^{-\sqrt{s/2}\,x}\right) + 3\cos 2\pi x$

$\frac{d\hat{u}}{dx}(3,s) = 0 \Rightarrow \sqrt{s/2}\,A_1\left(e^{3\sqrt{s/2}} - e^{3\sqrt{s/2}}\right) - 6\sin 2\pi$

$\Rightarrow A_1\left(e^{3\sqrt{s/2}} - e^{-3\sqrt{s/2}}\right) = 0 \Rightarrow A_1 = 0$

$u_p = \frac{1}{2D^2 - s}\,3\cos 2\pi x \Rightarrow \frac{3\cos 2\pi x}{2(-4\pi^2) - s} = \frac{3\cos 2\pi x}{s + 8\pi^2}$

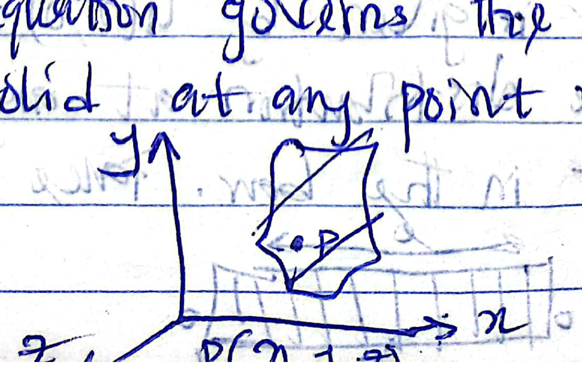
Solution of some classical PDE

Classical PDEs are differential equations - applied in physical processes. These equations - include: Heat diffusion, wave equations, Laplace equations etc.

Heat diffusion equation

The equation governs the heat distribution in a solid at any point and any time t.

~~tP~~



If $T(x,y,z,t)$ is the heat distribution in a solid at any point $P(x,y,z)$ and any time t, then the distribution is governed (determined) by the equation

$$\partial T/\partial t = \alpha^2 \nabla^2 T \Rightarrow \partial T/\partial t = \alpha^2\left[\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \frac{\partial^2 T}{\partial z^2}\right]$$

where $\alpha^2 = k/s\rho$

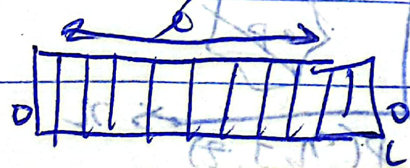
$k \rightarrow$ thermal conductivity
$s \rightarrow$ specific heat capacity
$\rho \rightarrow$ the density per unit volume

Note that:
If T is independent of y & z, then the equation reduces to:

$$\partial T/\partial t = \alpha^2 \partial^2 T/\partial x^2 \Rightarrow \text{1 dimensional heat equation}$$

One dimensional heat equation

Consider a thin bar of uniform density and length l insulated at both ends and kept at zero temperature. If the initial temperature in the bar is given by $u(x,0) = f(x)$. Find the temperature distribution at any point and any instant t in the bar. Take $\alpha^2 = 2$



According to heat distribution is governed by the equation $\partial T/\partial t = 2 \cdot \partial^2 T/\partial x^2, \quad t \geq 0, \; 0 \leq x \leq l$

$T(0,t) = 0, \; T(l,t) = 0, \; T(x,0) = f(x)$

Let $T(x,t) = X(x)T(t) \Rightarrow XT' = 2X''T$

or $\dfrac{X''}{X} = \dfrac{T'}{2T} = -\alpha^2$

$\therefore X'' + \alpha^2 X = 0$ —— (1)

$T' + 2\alpha^2 T = 0$ —— (2)

from equa (2) $T(t) = A_1 e^{-2\alpha^2 t}$

$X(x) = B_1 \cos \alpha x + B_2 \sin \alpha x$

$\therefore T(x,t) = e^{-2\alpha^2 t}[A\cos\alpha x + B\sin\alpha x]$

$\therefore T(0,t) = 0 \Rightarrow e^{-2\alpha^2 t}[A \cdot 1 + B \cdot 0] = 0 \Rightarrow A = 0$

$T(x,t) = Be^{-2\alpha^2 t} \sin \alpha x$

$T(x,t) = 0 \Rightarrow Be^{-2\alpha^2 t} \sin \alpha L = 0 \Rightarrow \sin \alpha L = 0$

or $\alpha L = n\pi \Rightarrow \alpha = n\pi/L$

$T(x,t) = \sum_{n=0}^{\infty} B_n e^{-2\frac{n^2\pi^2}{2}t} \sin \dfrac{n\pi}{L} x$

Again, $T(x,0) = f(x) \Rightarrow f(x) = \sum_{n=0}^{\infty} B_n \sin \dfrac{n\pi}{L} x$

$B_n = \dfrac{2}{L}\int_0^L f(x) \sin \dfrac{n\pi}{L} x \, dx$

Hence,

$T(x,t) = \sum_{n=0}^{\infty} \left[\dfrac{2}{L}\int_0^L f(x) \sin \dfrac{n\pi}{L} x \, dx\right] e^{-\frac{2n^2\pi^2 t}{L^2}} \sin \dfrac{n\pi}{L} x$

Example

Find the temperature distribution in a uniform thin bar of length 6cm, whose ends are insulated and kept at zero temperatures. If the initial temperature in the bar is given by $f(x) = 2x$, $0 \le x \le 6$. Take the diffusion by constant $\alpha^2 = 2$.

Soln.

The temperature distribution $T(x,t)$ is governed by the equation: $\partial T/\partial t = 2\,\partial^2 T/\partial x^2$, $t \ge 0$, $0 \le x \le t$. $T(0,t) = 0$, $T(6,t) = 0$, $T(x,0) = 2x$.

Let $T(x,t) = X(x) \cdot T(t) \Rightarrow XT' = 2X''T$

$\therefore \dfrac{X''}{X} = \dfrac{T'}{2T} = -\alpha^2$

$\Rightarrow X'' + \alpha^2 \cancel{X} = 0$ — ①

$T' + 2\alpha^2 T = 0$ — ②

$T(x,t) = e^{-2\alpha^2}[A\cos\alpha x + B\sin\alpha x]$

$T(0,t) = 0 \Rightarrow A = 0$

$\therefore T(x,t) = B\sin\alpha x\, e^{-2\alpha^2 t}$

$T(6,t) = 0 \Rightarrow B\sin 6\alpha\, e^{-2\alpha^2 t} \Rightarrow \sin 6\alpha = 0$.

$\therefore 6\alpha = n\pi$, $\alpha = n\pi/6$, $n = 0,1,2,\ldots$

$T(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{n^2\pi^2 t}{36}} \sin\dfrac{n\pi x}{6}$

$$T(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{n^2\pi^2 t}{18}} \sin\frac{n\pi x}{6}$$

$$T(x,0) = 2x = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi}{6}x$$

$$\Rightarrow B_n = \frac{2}{6}\int_0^6 2x \sin\frac{n\pi}{6}\,dx$$

$$B_n = \frac{2}{3}\int_0^6 x\sin\frac{n\pi x}{6}\,dx = \frac{2}{3}\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6} + \frac{6}{n\pi}\left[\int_0^6 \cos\frac{n\pi x}{6}\,dx\right]\right]$$

$$= \frac{2}{3}\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\Big|_0^6 + \frac{36}{n^2\pi^2}\sin\frac{n\pi x}{6}\Big|_0^6\right]$$

$$= -\frac{24}{n\pi}\cos n\pi$$

$$T(x,t) = \sum_{n=1}^{\infty} -\frac{24}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2 t}{18}} \sin\frac{n\pi x}{6} \quad \text{or}$$

$$T(x,t) = -\frac{24}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} e^{-\frac{n^2\pi^2 t}{18}} \sin\frac{n\pi}{6}x$$

$$T(x,t) = \frac{24}{\pi}\left(\frac{e^{-\frac{\pi^2 t}{18}}\sin\frac{\pi}{6}x}{1} - \frac{e^{-\frac{2\pi^2}{9}t}\sin\frac{\pi x}{3}}{2} + \frac{e^{-\frac{\pi^2 t}{2}}\sin\frac{\pi x}{2}}{3} + \cdots\right)$$

Exercises.

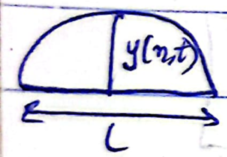
Find the temperature in a uniform (tin) bar 5m long whose ends are kept at zero temperatures, if the initial temperature in the bar is kept at $f(x) = 5(x-5)$. (Take $\alpha^2 = 3$).

2. Vibrating of a string.

Consider a flexible string, stretched along the x-axis and fixed at both ends. If the string is suddenly released from rest it will oscillate. If the initial velocity is zero. Then the displacement of the string along the y-axis is governed by the equation :

$$\partial^2 y/\partial t^2 = \alpha^2 \partial^2 y/\partial x^2 , \therefore (\alpha^2 = T/\rho)$$

where T is the tension in the string and ρ is the density (mass per unit length).



BC: $\partial^2 y/\partial t^2 = \alpha^2 \partial^2 y/\partial x^2 , \; y(0,t)=0, \; y(L,t)=0,$
$\partial y/\partial t (x,0)=0 , \; y(x,0)=f(x), \; 0 \le x \le L, \; t \ge 0.$

$\therefore$ let $y(x,t) = XT \Rightarrow XT'' = \alpha^2 X'' T$

$$\frac{T''}{\alpha^2 T} = \frac{X''}{X} = -c^2$$

$\Rightarrow X'' + c^2 X = 0$ —— ①

$T'' + \alpha^2 c^2 T = 0$ —— ②

From ① $X(x) = A_1 \cos cx + A_2 \sin cx$

From ② $T(t) = B_1 \cos c\alpha t + B_2 \sin \alpha c t$

$y(x,t) = (B_1 \cos \alpha c t + B_2 \sin \alpha c t)(A_1 \cos cx + A_2 \sin cx)$

$y(0,t) = 0 \Rightarrow (B_1 \cos \alpha c t + B_2 \sin \alpha c t)(A_1 \cdot 1 + A_2 \cdot 0)$

$\Rightarrow A_1 = 0$

$y(x,t) = A_2 \sin cx (B_1 \cos \alpha c t + B_2 \sin \alpha c t)$ or

$y(x,t) = \sin cx (A \cos \alpha c t + B \sin \alpha c t)$

$\partial y/\partial t = \sin cx\,(\alpha cA \sin\alpha ct + \alpha cB \cos\alpha ct)$

$\therefore \partial y/\partial t\,(x,0) = 0 \Rightarrow \sin cx\,(-\alpha cA \sin\alpha c.0 + \alpha cB \cos 0 = 0)$

$\Rightarrow B = 0$

$y(x,t) = A\cos\alpha ct \sin cx$

$y(l,t) = 0 \Rightarrow A\cos\alpha ct \sin cL = 0$

$\therefore \sin cL = 0$ or $cL = n\pi \Rightarrow c = \frac{n\pi}{L}$, $n = 0,1,2,\ldots$

$$y(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{\alpha n\pi t}{L} \sin\frac{n\pi x}{L}$$

$$y(x,0) = f(x) \Rightarrow f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{\alpha n\pi t}{L} \sin\frac{n\pi x}{L}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{L}x$$

where $A_n = 2/L \int_0^L f(x) \sin\frac{n\pi}{L}x\,dx$

$$y(x,t) = \sum_{n=0}^{\infty} \left[2/L \int_0^L f(x) \sin\frac{n\pi}{L}x\,dx \right] \cos\frac{\alpha n\pi t}{L} \sin\frac{n\pi x}{L}$$

Example:

If a flexible string (of uniform density) stretched and fixed at both ends, 4m apart is suddenly released and allowed to oscillate along the x-axis. Find the displacement along the y-axis $y(x,t)$ if the initial velocity is zero and the initial ~~tem-perature~~ displacement $f(x) = \frac{x-2}{4}$ (Take $\alpha^2 = \frac{T}{\rho} = 4$)

$y(x,0) = f(x)$.

Soln

$\partial^2 y/\partial t^2 = 4\,\partial^2 y/\partial x^2$, $y(0,t) = y(4,t) = 0$

$\partial y/\partial t\,(x,0) = 0$, $y(x,0) = f(x) = \frac{x-2}{4}$, $0 \le x \le 4$, $t \ge$

Let $y(x,t) = XT$

$$XT'' = 4X''T$$

$$\frac{T''}{4T} = \frac{X''}{X} = -c^2$$

$$\Rightarrow T'' + 4c^2 T = 0 \quad \text{--- (1)}$$

$$X'' + c^2 X = 0 \quad \text{--- (2)}$$

$T(t) = A_1 \cos 2ct + A_2 \sin 2ct$

$X(t) = B_1 \cos cx + B_2 \sin cx$

$y(x,t) = (A_1 \cos 2ct + A_2 \sin 2ct)(B_1 \cos cx + B_2 \sin cx)$

$y(0,t) = 0 \Rightarrow B_1 = 0$

$\therefore y(x,t) = \sin cx\,(A \cos 2ct + B \sin 2ct)$

$\partial y/\partial t\,(x,0) = 0 \Rightarrow B = 0$

$y(x,t) = A \sin cx \cdot \cos 2ct$

$y(4,t) = 0 \Rightarrow A \sin 4c \cos 2ct = 0$.

$\therefore \sin 4c = 0$ or $4c = n\pi \Rightarrow c = n\pi/4$

$$y(x,t) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi}{4} x \cos \frac{n\pi t}{2}$$

$$y(x,0) = \frac{x-2}{4} = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{4}.$$

$$\Rightarrow A_n = \frac{2}{4}\int_0^4 \left(\frac{x-2}{4}\right) \sin \frac{n\pi}{4} x \, dx$$

$$A_n = 1/8\int_0^4 (x-2)\sin\frac{n\pi}{4}x\,dx$$

$$= 1/8\int_0^4 x\sin\frac{n\pi}{4}x\,dx - 1/4\int_0^4 \sin\frac{n\pi}{4}x\,dx$$

$$= 1/8\left[-\frac{4x}{n\pi}\cos\frac{n\pi x}{4}\Big|_0^4 + \frac{4}{n\pi}\int_0^4 \cos\frac{n\pi}{4}x\,dx - 1/4\left(\frac{-4}{n\pi}\cos\frac{n\pi x}{4}\Big|_0^4\right)\right]$$

$$= -\frac{2\cos n\pi}{n\pi} + \frac{4^2}{n^2\pi^2}\sin\frac{n\pi x}{4}\Big|_0^4 + \frac{\cos n\pi - 1}{n\pi}$$

$$A_n = -\frac{2\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} = -\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}$$

$$= -\frac{\cos n\pi + 1}{n\pi}$$

$$y(x,t) = \sum_{n=1}^{\infty} -\left(\frac{1+\cos n\pi}{n\pi}\right)\sin\frac{n\pi x}{4}\cos\frac{n\pi t}{2}$$

Exercise

If a flexible string (of uniform density) stretched and fixed at both ends, 5m apart is suddenly released and allowed to oscillate along the x-axis. Find the displacement along the y-axis $y(x,t)$. If the initial velocity is zero and the initial displacement $f(x) = x^2/4$ (Take $a^2 = T/\rho = 4$)

$y(x,0) = f(x)$.

Laplace equation

This equation governs the steady state (time independent physical processes). It is an equation of the form: $\nabla^2 u = 0$, where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

Examples:

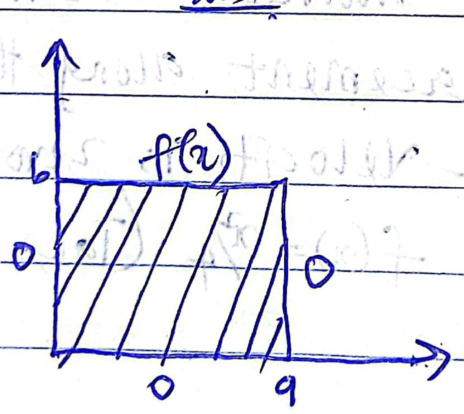
1. $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ 2. $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$

15/08/2023

Example

1. Find the steady state temperature distribution in a rectangular plate; $0 \le x \le a$, $0 \le y \le b$, whose edges are insulated and kept at zero temperature. If the other edges is maintained at a temperature given by $f(x)$ as follows: $u(0,y)$; $u(x,0) = 0$; $u(a,y) = 0$, $u(x,b) = f(x)$, $0 \le x \le a$, $0 \le y \le b$

Soln



Let $u(x,y)$ be the temperature in the plate. The temperature distribution is given by the Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

$u(0,y)=0,\ u(x,0)=0,\ u(a,y)=0,\ u(x,b)=f(x).$

Let $u(x,y)=X(x)Y(y) \Rightarrow X''Y+XY''=0$

$\Rightarrow X''/X=-Y''/Y=-\alpha^2$

$\Rightarrow X''+\alpha^2X=0$ —— (1)

$\Rightarrow Y''-\alpha^2Y=0$ —— (2)

$\therefore X(x)=A_1\cos\alpha x+A_2\sin\alpha x$

$Y(y)=B_1\cosh\alpha y+B_2\sinh\alpha y$

$u(x,y)=(A_1\cos\alpha x+A_2\sin\alpha x)(B_1\cosh\alpha y+B_2\sinh\alpha y)$

$u(0,y)=0 \Rightarrow (A_1\cdot 1+A_2\cdot 0)(B_1\cosh\alpha y+B_2\sinh\alpha y)=0$

$\Rightarrow A_1=0$

$\therefore u(x,y)=\sin\alpha x\,(A\cosh\alpha y+B\sinh\alpha y)$

$u(x,0)=0 \Rightarrow \sin\alpha x\,(A\cdot 1+B\cdot 0)=0 \Rightarrow A=0$

$\therefore u(x,y)=B\sin\alpha x\sinh\alpha y$

$u(a,y)=0 \Rightarrow B\sin\alpha a\sinh\alpha y\neq 0$

$\Rightarrow \sin\alpha a=0 \Rightarrow \alpha a=n\pi \Rightarrow \alpha=n\pi/a,\ n=1,2,\dots,n$

$u(x,y)=\sum_{n=0}^{\infty}B_n\sin\frac{n\pi}{a}x\,\sinh\frac{n\pi}{a}y$

$u(x,y)=f(x) \Rightarrow f(x)=\sum_{n=0}^{\infty}B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}b$

or $f(x)=\sum_{n=0}^{\infty}C_n\sin\frac{n\pi}{a}x,\ C_n=B_n\sinh\frac{n\pi}{a}b.$

$\cancel{u(x,y)=\sum_{n=0}^{\infty}2/a\int_0^a f(x)}$

$\Rightarrow C_n=2/a\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$

$$u(x,y) = \sum_{n=0}^{\infty} \frac{\left[2/a \int_0^a f(x) \sin \frac{n\pi}{a}x \, dx \right]}{\sinh \frac{n\pi}{a} b} \sin \frac{n\pi}{a}x \sinh \frac{n\pi}{a} y$$

② Find the steady state temperature distribution in a rectangular plate $0 \le x \le 2$, $0 \le y \le 3$, if the upper edge is kept at a temperature $80x$ and the other edges are maintained at zero temperatures.

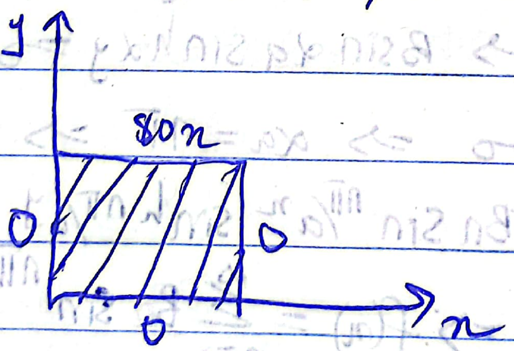
Soln.

Let $T(x,y)$ be the steady state temperature, then the temperature distribution satisfies the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 \le x \le 2, \quad 0 \le y \le 3$$

~~$T(x,0)=0$, $T(x,3)=80x$,~~

$T(0,y)=0$, $T(x,0)=0$, $T(x,3)=80x$, $T(2,y)=0$.



Let $T(x,y) = X(x)Y(y) \Rightarrow X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2$$

$\Rightarrow X'' + \alpha^2 X = 0$ —— ①

$Y'' - \alpha^2 Y = 0$ —— ②

from ① $X(x) = A_1 \cos \alpha x + A_2 \sinh \alpha x$

from ② $Y(y) = B_1 \cosh \alpha y + B_2 \sinh \alpha y$

$T(x,y) = (A_1 \cos \alpha x + A_2 \sinh \alpha x)(B_1 \cosh \alpha y + B_2 \sinh \alpha y)$

$T(0,y) = 0 \Rightarrow (A_1 \cdot 1 + A_2 \cdot 0)(B_1 \cosh \alpha y + B_2 \sinh \alpha y)$

$\Rightarrow A_1 = 0$

$T(x,y) = \sin \alpha x (A \cosh \alpha y + B \sinh \alpha y)$

$T(x,0) = 0 \Rightarrow \sin \alpha x (A \cdot 1 + B \cdot 0) = 0 \Rightarrow A = 0$

$T(x,y) = B \sin \alpha x \sinh \alpha y$

$T(2,y) = 0 \Rightarrow B \sin 2\alpha \sinh \alpha y = 0 \Rightarrow \sin 2\alpha = 0$

$\Rightarrow 2\alpha = n\pi \Rightarrow \alpha = n\pi/2, \; n = 0, 1, 2, \ldots$

$T(x,y) = \sum_{n=0} B_n \sin \frac{n\pi}{2} x \sinh \frac{n\pi}{2} y$

$T(x,3) = 80x \Rightarrow 80x = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{2} x \sinh \frac{3n\pi}{2}$

or

$80x = \sum_{n=0}^{\infty} C_n \sin \frac{n\pi}{2} x \; ; \; C_n = B_n \sinh 3n\pi/2$

$C_n = \frac{2}{2} \int_0^2 80x \sin \frac{n\pi}{2} x \, dx$

$= 80 \left[-\frac{2x}{n\pi} \cos \frac{n\pi}{2} x \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi}{2} x \, dx \right.$

$= \frac{160}{n\pi} \left[2 \cos n\pi \right] = \frac{-360}{n\pi} \cos n\pi.$

$\therefore C_n = -\frac{320}{n\pi} \cos n\pi = B_n \sinh \frac{3n\pi}{2} \Rightarrow B_n = \frac{-320 \cos n\pi}{n\pi \sinh \frac{3n\pi}{2}}$

$T(x,y) = -\frac{320}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n \sinh \frac{3n\pi}{2}} \cdot \sin \frac{n\pi}{2} x \sinh \frac{n\pi}{2} y$

Ex:

① Solve the two dimensional Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ given that $u(0,y) = u(x,0) = 0$, $u(x,2) = 0$, $u(3,y) = 2y-1$, $0 \le x \le 3$, $0 \le y \le 2$

② Solve the Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, $u(x,0) = 0$, $u(0,y) = 0$, $\partial u/\partial x(2,y) = 0$, $u(x,1) = x l\ldots$ $0 \le x \le 2$, $0 < y \le 3$.

22/08/2023

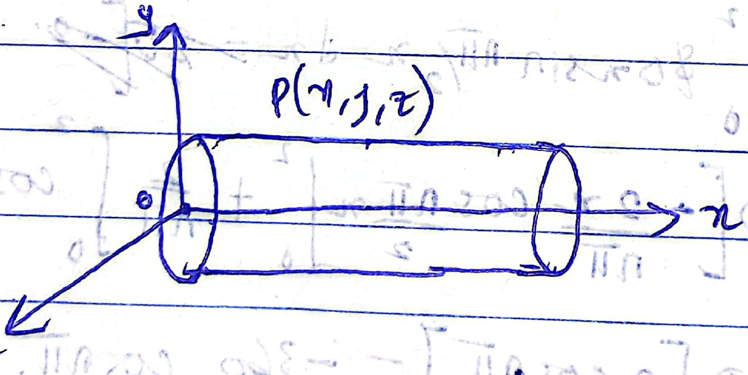
Laplace equation

$$\nabla^2 u = 0 \Rightarrow \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0.$$

Cylindrical coordinate

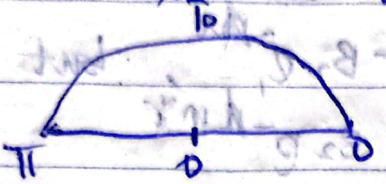
In cylindrical coordinate the Laplace equation is expressed in the form

$$\nabla^2 u = 1/r \, \partial/\partial r (r \, \partial u/\partial r) + 1/r^2 \, \partial^2 u/\partial \theta^2 + \partial^2 u/\partial z^2 = 0$$



Example

Find the steady state temperature distribution (T) at any point in a semi circlar metal plate of radius a units whose temperature at the circumference is maintained at a given temperature T_0 where as the base temperature is kept at zero.



Note that the laplace equation for the temperature distribution is $\partial^2 T/\partial r^2 + 1/r \, \partial T/\partial r + 1/r^2 \, \partial^2 T/\partial \theta^2 = 0$.

$T(0,\theta) = 0$, $T(r,\pi) = 0$, $T(a,\theta) = T_0$, $T(r,0) = 0$

From the above equation $\partial^2 T/\partial r^2 + 1/r \, \partial T/\partial r + 1/r^2 \, \partial^2 T/\partial \theta^2 = 0$

$$r^2 \, \partial^2 T/\partial r^2 + r \, \partial T/\partial r + \partial^2 T/\partial \theta^2 = 0$$

Let $T(r,\theta) = R(r)\Theta(\theta)$

$\Rightarrow r^2 R'' \Theta + r R' \Theta + R \Theta'' = 0$

Dividing through by $R\Theta$

$$r^2 \frac{R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$\Rightarrow r^2 R'' + rR' - \lambda^2 R = 0$ ——— ①

$\Theta'' + \lambda^2 \Theta = 0$ ——— ②

from ② $\Theta(\theta) = A_1 \cos \lambda\theta + A_2 \sin \lambda \theta$

From ①

Let $r = e^t \Rightarrow d/dr = d/dt \times dt/dr = e^{-t} \, d/dt$

$d^2/dr^2 = d/dr(d/dr) = d/dt \, [dr/dt \, (e^{-t} \, d/dt)$

$= e^{-t} \, d/dt(e^{-t} \, d/dt) = e^{-2t}(d^2/dt^2 - d/dt)$

substitute in to equation ① gives.

$e^{2t} \cdot e^{-2t}(d^2R/dt^2 - dR/dt) + e^t \cdot e^{-t} \, dR/dt - \lambda^2 R = 0$

$$\Rightarrow \frac{d^2R}{dt^2} - \frac{dR}{dt} + \frac{dR}{dt} - \lambda^2 R = 0$$

$$\therefore \frac{d^2R}{dt^2} - \lambda^2 R(t) = 0 \Rightarrow m^2 - \lambda^2 = 0 \Rightarrow m = \pm \lambda i$$

$$\Rightarrow R(t) = B_1 e^{\lambda t} + B_2 e^{-\lambda t}, \quad \text{but } t = \ln r$$

$$R(r) = B_1 e^{\lambda \ln r} + B_2 e^{-\lambda \ln r}$$

$$R(r) = B_1 r^\lambda + B_2/r^\lambda$$

$$\therefore T(r,\theta) = (B_1 r^\lambda + B_2/r^\lambda)(A_1 \cos\lambda\theta + A_2 \sin\lambda\theta)$$

since $T(r,\theta)$ is bdd when $r=0 \Rightarrow B_2 = 0$

$$T(r,\theta) = B_1 r^\lambda (A_1 \cos\lambda\theta + A_2 \sin\lambda\theta)$$

$$= r^\lambda (A\cos\lambda\theta + B\sin\lambda\theta)$$

$$T(r,0) = 0 \Rightarrow r^\lambda (A\cos\lambda\cdot 0 + B\sin\lambda\cdot 0) = 0$$

$$\Rightarrow r^\lambda (A\cdot 1 + B\cdot 0) = 0 \Rightarrow A = 0$$

$$T(r,\theta) = Br^\lambda \sin\lambda\theta$$

$$T(r,\pi) = 0 \Rightarrow Br^\lambda \sin\lambda\pi = 0 \Rightarrow \sin\lambda\pi = 0$$

$$\Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$$

$$T(r,\theta) = B_n r^n \sin n\theta$$

By principle of superposition

$$T(r,\theta) = \sum_{n=0}^{\infty} B_n r^n \sin n\theta$$

$$T(a,\theta) = T_0 = \sum_{n=0}^{\infty} B_n a^n \sin n\theta = \sum_{n=0}^{\infty} C_n \sin n\theta$$

$$\Rightarrow C_n = B_n a^n$$

$$T_0 = \sum_{n=0}^{\infty} C_n \sin n\theta \quad \text{where } C_n = \frac{2}{\pi}\int_0^\pi T_0 \sin n\theta \, d\theta$$

$$= \frac{2}{\pi} T_0 \left(\frac{-\cos n\theta}{n} \Big|_0^\pi \right)$$

$$C_n = -\frac{2T_0}{\pi} \frac{(\cos n\pi - 1)}{n} \; \cancel{\text{But}}$$

But $C_n = B_n a^n \Rightarrow B_n = C_n / a^n$

$$\therefore B_n = -\frac{2T_0}{n\pi}\left(\frac{\cos n\pi - 1}{a^n}\right)$$

$$T(r,\theta) = \sum_{n=1}^{\infty} \frac{2T_0 (1-\cos n\pi)}{n\pi\, a^n} r^n \sin n\theta$$

$$T(r,\theta) = \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (1-\cos n\pi) \sin n\theta.$$

Exercises.

(1). A circular plate of unit radius whose faces are insulated has half of its boundary kept at constant temperature u_1 and the other half at constant temperature u_2. Find the steady state temperature of the plate.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

$$T(1,\theta) = \begin{cases} u_1, & 0 < \theta < \pi \\ u_2, & \pi < \theta < 2\pi \end{cases}$$

(2) A square plate having sides 3cm has its edges fixed in the xy plane and is set into transverse vibration. If the displacement $z(x,y,t)$ at any point $p(x,y)$ and at any time t is governed by $\partial^2 z/\partial t^2 = c^2(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2)$ where c^2 is a constant. Find $z(x,y,t)$ if the plate is given an initial shape $f(x,y)$ and released with initial velocity

Topic: Linear Partial Differential Equation:

C.A solution

Q.1.6. solve $\partial^2 y/\partial t^2 = 4\, \partial^2 y/\partial x^2$, $y(x,0) = 0$, $y(0,t) = y(\pi, t) = 0$, $\partial y/\partial t(x,0) = 3\sin x - 2\sin 5x$.

soln.

Let $y(x,t) = XT \Rightarrow XT'' = 4X''T$

$T''/4T = X''/X = -\alpha^2$

$T'' + 4\alpha^2 T = 0$ ——— (1)

$X'' + \alpha^2 X = 0$ ——— (2)

$T(t) = A_1 \cos 2\alpha t + A_2 \sin 2\alpha t$

$X(x) = B_1 \cos\alpha x + B_2 \sin\alpha x$

$y(x,t) = (A_1 \cos 2\alpha t + A_2 \sin 2\alpha t)(B_1 \cos\alpha x + B_2 \sin)$

$y(x,0) = 0 \Rightarrow A_1 = 0$

$y(x,t) = \sin 2\alpha t\,(A\cos\alpha x + B\sin\alpha x)$

$y(0,t) = 0 \Rightarrow A = 0$

$y(x,t) =$

Ans

$y(x,t) = 3/2 \sin x \sin 2t - 1/5 \sin 5x \sin 10t$ //

Q.2. Use Laplace transform $\partial u/\partial x = 2\frac{\partial u}{\partial t} + u$,

$u(x,0) = 6e^{-3x}$, $u(0,t) = 6e^{-2t}$

Let $\mathcal{L}\, u(x,t) = \int_0^\infty e^{-st} u(x,t)\, dt = U(x,s)$

Taking the Laplace transform of both side w.r.t

$dy/dx + Py = Q$

$\mathcal{L}\, \partial u/\partial x = 2\mathcal{L}\, \partial u/\partial t + \mathcal{L}u \Rightarrow \frac{d\hat{u}}{dx} = \frac{d\hat{u}}{dx} = 2s\hat{u} - 2u(x,0)$

$\frac{d\hat{u}}{dx} - (2s+1)\hat{u} = 2u(x,0)$

$\Rightarrow \frac{d\hat{u}}{dx} - (2s+1)\hat{u} = -12e^{-3x}$

$\mathcal{L}u(0,t) = \mathcal{L}\,6e^{-2t} \Rightarrow \hat{u}(0,s) = 6/s+2$

$\Rightarrow \hat{u}e^{-(1+2s)x} = \int -12e^{-3x}\, e^{-(1+2s)} dx + c$

$ye^{\int P dx} = \int Q e^{\int P dx} dx + c$

$\Rightarrow \hat{u}\, e^{-(2s+1)x} = -12\int e^{-2(s+2)x} dx + c = \frac{-12e^{-2(s+2)x}}{-(2s+4)}$

$\hat{u}e^{-(2s+1)x} = 6e^{-2(s+2)x} + c$

$\hat{u}(0,s) = \frac{6}{s+2}e^{-3x} + ce^{(2s+1)x}$

$\hat{u}(0,s) = 6/s+2 \Rightarrow 6/s+2 + c\cdot 1 \Rightarrow c = 6/s+2 - 6/s+2 = 0$

$\Rightarrow c = 0$

$\bar{u}(x,s) = 6/s+2\; e^{-3x} \Rightarrow u(x,t) = \mathcal{L}^{-1}\hat{u}(x,s)$

i.e. $u(x,t) = \mathcal{L}^{-1}\frac{6}{s+2}e^{-3x} = e^{-3x}\mathcal{L}^{-1}\, 6/s+2$

$u(x,t) = e^{-3x}\cdot 6e^{-2t} = 6e^{-(3x+2t)}$ //

Q. 26

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\frac{2n^2\pi^2 t}{9}} \cos \frac{n\pi}{3} x$$

$$A_n = A_0/2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{3} x \quad , \quad A_0 = 2/3 \int_0^3 3x \, dx = 9$$

$$A_n = -9/n^2\pi^2 (\cos n\pi - 1) \Rightarrow A_n = \frac{18}{n^2\pi^2} (1 - \cos n\pi)$$

Ans

$$u(x,t) = 9/2 + \frac{18}{n^2} \sum_{n=0}^{\infty} \frac{(1 - \cos n\pi)}{\pi^2} e^{-\frac{2n^2\pi^2 t}{9}} \cos \frac{n\pi}{3} x$$

TOPIC First order Linear partial differential equation

The general form of a first order linear PDE is given. $P\partial z/\partial x + Q\partial z/\partial y = R$, where P, Q are functions of x and y.

$Pp + Qq = R$, $p = \partial z/\partial x$, $q = \partial z/\partial y$

Equation (*) can be solved by reducing to the canonical form called the Lagrange's form.

$dx/P = dy/Q = dz/R$

Using combination of any two e.g $dx/P = dy/Q$ or $dx/P = dz/R$. primitive ~~of the~~ function of the $u(x,y,z) = c$, and $v(x,y,z) = c_2$ are obtained.

Hence the general solution of the partial diff equation is given by $\phi(u,v) = 0$.

Example.

1. Solve the PDE i. $2\partial z/\partial x + 3\partial z/\partial y = 1$

$P = 2$, $Q = 3$, $R = 1$.

The Lagrange's equation are

$dx/2 = dy/3 = dz/1 \Rightarrow dx/2 = dy/3 \Rightarrow 3dx - 2dy = 0$

integrate

$3x - 2y = c_1 \Rightarrow u = 3x - 2y = c_1$

again, $dx/2 = dz/1 \Rightarrow dx = 2dz \Rightarrow x - 2z = c_2$

$v = x - 2z = c_2$. ∴ The general solution to the PDE is

$\phi(3x - 2y, x - 2z) = 0$.

2. $x\,\partial z/\partial x + y\,\partial z/\partial y = 3z$

$P = x,\ Q = y,\ R = 3z$

The xtics equation are

$$dx/x = dy/y = dz/3z$$

$\Rightarrow dx/x = dy/y \Rightarrow$ integrate both sides gives

$\ln x - \ln y = \ln k \Rightarrow \ln(x/y) = \ln k \Rightarrow x/y = k$ — ①

$dx/x = dz/3z \Rightarrow 3\,dx/x = \frac{dz}{z} \Rightarrow 3\ln x - \ln z = \ln c$

$\ln(x^3/z) = \ln c \Rightarrow x^3/z = c$ or $\frac{z}{x^3} = k_2$

The general solution is $\phi(x/y,\ z/x^3) = 0$

3. $y^2 z\,\partial z/\partial x - x^2 z\,\partial z/\partial y = x^2 y$

$P = y^2 z,\ Q = -x^2 z,\ R = x^2 y$

$$\Rightarrow \frac{dx}{y^2 z} = -\frac{dy}{x^2 z} = \frac{dz}{x^2 y}$$

$$\Rightarrow \frac{dx}{y^2 z} = -\frac{dy}{x^2 z} \Rightarrow x^2\,dx + y^2\,dy = c$$

$x^3/3 + y^3/3 = c \Rightarrow x^3 + y^3 = c_1$

$-dy/x^2 z = \frac{dz}{x^2 y} \Rightarrow y\,dy + z\,dz = 0$

$y^2/2 + z^2/2 = k \Rightarrow y^2 + z^2 = c_2$

$\therefore \phi(u, v) = 0 \Rightarrow \phi(y^2 + z^2,\ x^3 + y^3) = 0.$

Second order linear PDE

The general form of a second order PDE is given by $A\partial^2u/\partial x^2 + 2B\partial^2u/\partial x\partial y + C\partial^2u/\partial y^2 = f(u_x, u_y, x, y, u)$

The above equation is classified on

i. Parabolic if $B^2 - AC = 0$

ii. Elliptic if $B^2 - AC < 0$

iii. Hyperbolic if $B^2 - AC > 0$

Classify the following equations into parabolic, elliptical or hyperbolic

i. $\partial^2u/\partial x^2 + 2\partial^2u/\partial x\partial y + 3\partial^2u/\partial y^2 = 3u$

ii $x^2\frac{\partial^2u}{\partial x^2} + (1-y^2)\frac{\partial^2u}{\partial y^2} = 2xy$, $0 \le x \le 1$, $0 \le y < \infty$

Soln

1. $A = 1$, $B = 1$, $C = 3$

$B^2 - AC = 1 - 3 = -2 < 0 \Rightarrow$ Elliptic

ii. $A = x^2$, $B = 0$, $C = 1 - y^2$

$B^2 - AC = 0 - x^2(1-y^2) = x^2y^2 - x^2$

Exercise

Solve the PDE $y^2\partial z/\partial x - xy\,\partial z/\partial y = x(z-2y)$

$\boxed{\frac{dz}{dy} + \frac{z}{y} = 2}$

$dx/y^2 = dy/-xy = dz/x(z-2y)$

Classify the following:

1. $2\partial^2 u/\partial x^2 - 6\partial^2 u/\partial x\partial y + 3\partial^2 u/\partial y^2 + 6\partial u/\partial x + 3\partial u/\partial x^2 - xy = 0$

2. $x^2 \partial^2 u/\partial x^2 + 2xy\, \partial^2 u/\partial x\partial y + y^2 \partial^2 u/\partial y^2 = 0.$

11/09/2023

Lagrange's multiplier (Method).

Given the PDE: $P\,\partial z/\partial x + Q\,\partial z/\partial y = R$

With the characteristic equation $dx/P = dy/Q = R$.

Let l, m, n be constants or functions of x, y, z, then

$$dx/P = dy/Q = dz/R = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}.$$

the l, m and n are choosen such that $lP + mQ + nR =$

Thus the above equation reduces $l\,dx + m\,dy + n\,dz = 0$.

Accordingly the solution of the equation is obtained by integrating $l\,dx + m\,dy + n\,dz = 0$ or $u(x, y, z) = c_1$.

l, m and n are called Lagrange's multipliers.

Example

solve the equation $(y+z)p - (x+z)q = x - y$

Soln.

The characteristics equation(s) are: $\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$

Taking the multipliers (1, 1, 1)
Note that:
$1(y+z) - 1(x+z) + 1(x-y) = y+z-x-z+x-y = 0$

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{1dx+1dy+1dz}{1(y+z)-(x+z)+(x-y)}$$

$$= \frac{dx+dy+dz}{y+z-x-z+x-y}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

$\Rightarrow dx+dy+dz = 0 \Rightarrow x+y+z = c_1$

Similarly, using the multipliers: $(x, y, -z)$
Note that: $x(y+z) - y(x+z) - z(x-y)$
$= xy + xz - xy - yz - zx + zy = 0$

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{xdx+ydy-zdz}{0}$$

$\therefore xdx + ydy - zdz = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c$

$\therefore x^2 + y^2 - z^2 = c_2$

Hence, the solution to the given PDE is
$\phi(x+y+z,\ x^2+y^2-z^2) = 0$

(2) Solve the first order equation:

$$xz\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial y} = y^2 - x^2$$

Soln

The characteristics equation is $\frac{dx}{xz} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$

$\therefore \frac{dx}{xz} = \frac{dy}{-zy} \Rightarrow \frac{dx}{x} = -\frac{dy}{y} \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$

$\ln x + \ln y = c \Rightarrow \ln(xy) = c \Rightarrow xy = e^c$ or $xy = c_1$

Using the multipliers (x, y, z) we $x \cdot xz + y(-zy) + z(y^2-x^2)$

$= x^2z - zy^2 + zy^2 - zx^2 = 0$

Hence $\frac{dx}{xz} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0 \Rightarrow x^2 + y^2 + z^2 = c_2$.

Hence, the solution of the equation is

$\phi(xy, x^2+y^2+z^2) = 0$.

Exercise

Use the Lagrange multiplier method to solve the equation : $(y-z)p + (x-y)q = \cancel{z}\, x$! (Hint: use $(1,1,1)$).

Second order linear partial diff equ. with Constant coeff.
(Homogeneous)

Given the equation : $A \partial^2 z/\partial x^2 + B \partial^2 z/\partial x \partial y + C \partial^2 z/\partial y^2 = 0$ ⊛

where A, B and C are constant.

Let $z = \phi(y + mx) \Rightarrow z = \phi(u)$

$\therefore \partial z/\partial x = \partial \phi/\partial u \cdot \partial u/\partial x = m \, \partial\phi/\partial u$

$\partial z/\partial y = \partial\phi/\partial u \cdot \partial u/\partial y = 1 \, \partial\phi/\partial u$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \partial/\partial x \left(\partial/\partial x\right) = \partial/\partial x \left(m \, \partial\phi/\partial u\right) = \frac{d}{du} \frac{\partial u}{\partial x}\left(m \frac{\partial \phi}{\partial u}\right)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = m^2 \frac{d^2\phi}{du^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \partial/\partial x \left(\partial\phi/\partial u\right) = d/du \cdot \partial u/dx \left(d\phi/\partial u\right) = m \frac{d^2\phi}{du^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \partial/\partial y \left(\partial\phi/\partial u\right) = \frac{du}{du} \cdot \frac{\partial u}{\partial u}\left(d\phi/\partial u\right) = \frac{d^2\phi}{du^2}$$

$\therefore$ Substituting into the equation ⊛ given

$$Am^2 \frac{d^2\phi}{du^2} + Bm \frac{d^2\phi}{du^2} + C \frac{d^2\phi}{du^2} = 0$$

$\Rightarrow (Am^2 + Bm + C) \frac{d^2\phi}{du^2} = 0 \Rightarrow Am^2 + Bm + C = 0$ is the auxiliary equation

Note that

① If the roots are real and distinct m_1 and m_2, $m_1 \neq m_2$. Then the general solution of the equ. is $z(x, y) = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$

(2) If the roots are real and equal i.e $m = m_1 = m_2$ then the general solution of the equation is

$$z(x,y) = \phi_1(y + m_1 x) + x\phi_2(y + m_2 x)$$

(3) If the roots are complex i.e $m_1 = a+ib$, $m_2 = a-ib$, then

$$z(x,y) = \phi_1(y + (a+ib)x) + \phi_2(y + (a-ib)x)$$

Examples:

Solve the equation

(1) $\partial^2 z/\partial x^2 + 3\partial^2 z/\partial x \partial y + 2\partial^2 z/\partial y^2 = 0$

Let $z = \phi(y + mx)$

The auxiliary equation is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \Rightarrow m = -1, m = -2$

The general solution is

$$z(x,y) = \phi_1(y - x) + \phi_2(y - 2x)$$

(2) $\partial^2 z/\partial x^2 - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = 0$

The auxiliary equation is $m^2 - 4m + 4 = 0$

Thus, the roots are $m = 2, 2$.

Accordingly $z(x,y) = \phi_1(y + 2x) + x\phi_2(y + 2x)$.

(3) $\partial^2 z/\partial x^2 - 2\,\partial^2 z/\partial x \partial y + 4\,\partial^2 z/\partial y^2 = 0$

The a.e is $m^2 - 2m + 4 = 0$

$\therefore\ m = \frac{2 \pm \sqrt{4 - 4.4}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$

$\therefore\ m_1 = 1 + \sqrt{3}i,\ m_2 = 1 - \sqrt{3}i$

$z(x,y) = \phi_1(y + (1+\sqrt{3}i)x) + \phi_2(y + (1-\sqrt{3}i)x)$.

(4) $\partial^3 z/\partial x^3 + 3\,\partial^3 z/\partial x^2 \partial y - 4\,\partial^3 z/\partial y^3 = 0$.

The a.e is $m^3 + 3m^2 - 4 = 0$.

$\therefore (m-1)(m^2 + 4m + 4) = 0$

$m = 1,\ m = -2, -2$.

$z(x,y) = \phi_1(y + x) + \phi_2(y - 2x) + x\phi_3(y - 2x)$.

Exercise

Solve the following equation.

i. $\partial^2 z/\partial x^2 + \partial^2 z/\partial x \partial y - \partial^2 z/\partial y^2 = 0$

ii. $\partial^2 z/\partial x^2 - 5\,\partial^2 z/\partial x \partial y + 6\,\partial^2 z/\partial y^2 = 0$

Given that $\partial^2 u/\partial x \partial t = \sin x \cos t$

If $\frac{\partial u}{\partial x} = 2x$, at $t = \pi/2$ and

$u = 2\sin t$, at $x = \pi$

$\partial/\partial x (\partial u/\partial t) = \sin x \cos t$

$\partial u/\partial x = \sin x \sin t + c(x)$

at $t = \pi/2$, $\partial u/\partial x = 2x$

$2x = \sin x \sin \pi/2 + c(x)$

$2x = \sin x + c(x)$

$\therefore$ ~~$c(x) = \sin x + 2x$~~

~~$\partial u/\partial x = 2x + \sin x - \sin x \sin t$~~

~~$u(x,t) = x^2$~~

$c(x) = 2x - \sin x$

$\partial u/\partial x = \sin x \sin t + 2x - \sin x$

$u(x,t) = -\cos x \sin t + x^2 + \cos x$

at $x = \pi$, $u = 2\sin t$

$2\sin t = -\sin t + \pi^2 + 1 + D(y)$

$\therefore 3\sin t + \pi^2 + 1 = D(y)$

Q. $z = (x-a)^2 + (y-b)^2$

Diff. w.r.t x

$z_x = 2(x-a)$ —— (1)

$z_y = 2(y-b)$ —— (2)

$\therefore (x-a) = z_x/2$

$(y-b) = z_y/2$

$z = \left(\frac{z_x}{2}\right)^2 + \left(\frac{z_y}{2}\right)^2$

Q. $\partial^2 u/\partial x \partial y = 4e^y \cos 2x$, given that at $y=0$, $\partial u/\partial x = \cos x$ and at $x = \pi$, $u = y^2$.

sol^n

$\partial^2 u/\partial x \partial y = 4e^y \cos 2x$

$\partial/\partial y(\partial u/\partial x) = 4e^y \cos 2x$

integrate w.r.t y

$\partial u/\partial x = 4e^y \cos 2x + c(x)$

since $y=0$ at $\partial u/\partial x = \cos 2x$, then

$\cos 2x = 4\cos 2x + c(x)$

$\therefore c(x) = -3\cos 2x$

$\partial u/\partial x = 4e^y \cos 2x - 3\cos 2x$

integrate w.r.t x

$u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + D(y)$

$y^2 = D(y)$

$u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + y^2$

Q. $\partial^2/\partial x \partial t = \sin x \cos t$, if $\partial u/\partial x = 2x$ at $t = \pi/2$ and $u = 2\sin t$ at $x = \pi$.

sol^n

$\partial^2 u/\partial x \partial t = \sin x \cos t$

$\partial/\partial t(\partial u/\partial x) = \sin x \cos t$

integrate w.r.t t

$\partial u/\partial x = \sin x \sin t + c(x)$

at $t=0$, $\partial u/\partial x = 2x$

$2x = c(x) \Rightarrow c(x) = 2x$

$\partial u/\partial x = \sin x \sin t + 2x$

integrate w.r.t x

$u(x,t) = -\cos x \sin t + x^2 + D(y)$

$2\sin t = \sin t + x^2 + D(y)$.

$\therefore D(y) = \sin t - x^2$

$u(x,t) = -\cos x \sin t + x^2 - x^2 + \sin t$

$u(x,t) = \sin t - \cos x \sin t$.

Q. $\partial^2 z/\partial x \partial y = 8e^y \sin 2x$, if $\partial z/\partial x = \sin 2x$, and $z = 2y^2$ at $x = \pi/2$

sol^n

$\partial^2 z/\partial x \partial y = 8e^y \sin 2x$

$\partial/\partial y(\partial z/\partial x) = 8e^y \sin 2x$

integrate w.r.t y

$\partial z/\partial x = 8e^y \sin 2x + c(x)$

$\sin 2x = 8\sin 2x + c(x)$

$\therefore c(x) = -7\sin 2x$

$\partial z/\partial x = 8e^y \sin 2x - 7\sin 2x$

integrate w.r.t x

$z(x,y) = -4e^y \cos 2x + 7/2 \cos 2x + D(y)$

$2y^2 = 4e^y - 7/2 + D(y)$

$D(y) = 2y^2$

$z(x,y) = -4e\cos 2x + 7/2 \cos 2x + 2y^2$

Formation Diff equation

(1) $\phi(xy+z^3,\ y+z)=0$

let $u = xy+z^3$ \$ $v = y+z$

$\phi(u,v)$

Differentiate w.r.t x

$\partial\phi/\partial u \cdot \partial u/\partial x + \frac{\partial\phi}{\partial v}\cdot \partial v/\partial x$

$\partial\phi/\partial u(1\cdot y+3z^2 z_x)+\frac{\partial\phi}{\partial v}(z_x)=0$

$(y+3z^2 z_x)\,\partial\phi/\partial u + z_x\,\partial\phi/\partial v = 0$ — (1)

Differentiate w.r.t y

$\partial\phi/\partial u\cdot \partial u/\partial y + \partial\phi/\partial v\cdot\partial v/\partial y$

$\partial\phi/\partial u(x\cdot 1+3z^2 z_y)+\partial\phi/\partial v(z_y)=0$

$(x+3z^2 z_y)\,\partial\phi/\partial u + z_y\,\partial\phi/\partial v = 0$ — (2)

Now.

$$\begin{pmatrix} y+3z^2 z_x & z_x \\ x+3z^2 z_y & z_y \end{pmatrix}\begin{pmatrix}\partial\phi/\partial u \\ \partial\phi/\partial v\end{pmatrix} = 0$$

$$\begin{vmatrix} y+3z^2 z_x & z_x \\ x+3z^2 z_y & z_y \end{vmatrix} = 0$$

$y z_y + \cancel{3z^2 z_x z_y} - x z_x - \cancel{3z^2 z_x z_y} = 0$

$(y-x)z_x = 0$ is required differential equation.

(2) $\phi(x^2+y^2,\ xy-z)=0$

let $u = x^2+y^2$ \$ $v = xy-z$

$\phi(u,v)$

Differentiate w.r.t x

$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0$

$\frac{\partial\phi}{\partial u}(2x)+\frac{\partial\phi}{\partial v}(y-z_x)=0$

$2x\frac{\partial\phi}{\partial u}+(y-z_x)\frac{\partial\phi}{\partial v}=0$ — (1)

$\frac{\partial\phi}{\partial u}(2y)+\frac{\partial\phi}{\partial v}(x-z_y)=0$

$2y\,\partial\phi/\partial u+(x-z_y)\,\partial\phi/\partial v=0$ — (2)

Now

$$\begin{pmatrix} 2x & y-z_x \\ 2y & x-z_y \end{pmatrix}\begin{pmatrix}\partial\phi/\partial u \\ \partial\phi/\partial v\end{pmatrix}=0$$

$$\begin{vmatrix} 2x & y-z_x \\ 2y & x-z_y \end{vmatrix}=0$$

$2x^2-2xz_y-2y^2+2yz_x=0$

$2yz_x-2xz_y=2y^2-2x^2$

$yz_x-xz_y=y^2-x^2$ is the required solution of differential equation

(3) $\phi(z^2+xy,\ x+y+z)=0$

let $u=z^2+xy$ \$ $v=x+y+z$

$\phi(u,v)=0$

$\partial\phi/\partial u(2zz_x+1\cdot y)+\partial\phi/\partial v(1+z_x)=0$ — (1)

$\partial\phi/\partial u(2zz_y+1\cdot x)+\partial\phi/\partial v(1+z_y)=0$ — (2)

$$\begin{pmatrix} 2zz_x+y & 1+z_x \\ 2zz_y+x & 1+z_y \end{pmatrix}\begin{pmatrix}\partial\phi/\partial u \\ \partial\phi/\partial v\end{pmatrix}=0$$

$(2zz_x+y)(1+z_y)+(2zz_y+x)(1+z_x)=0$

$\cancel{2zz_x+2zz_xz_y+2zz_xz_y+y+yz_y+2zz_y+}$

$\cancel{2zz_yz_x+2(}$

$2z_x + 2z_x z_y + y + y z_y + 2\cdot z_y +$
$2z_y z_x + x + x z_x = 0$

$4z_x z_y + (2+x)z_x + (2+y)z_y = -x-y$
is the required differential equation

(4) $\phi(z/x^2, x-y) = 0$

Let $u = z/x^2$ & $v = x-y$

$\phi(u,v) = 0$

$\partial\phi/\partial u \left(\frac{x z_x - z}{x^2}\right) + \partial\phi/\partial v\,(1)$ — (1)

$\partial\phi/\partial u\,(x z_x) + \partial\phi/\partial v\,(-1)$ — (2)

$$\begin{pmatrix} \frac{x z_x - z}{x^2} & 1 \\ x z_x & -1 \end{pmatrix}\begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} = 0$$

$-\frac{x z_x + z}{x^2} - x z_x = 0$

∴ ~~$2z_x + z$~~ $(x^2 z_x - x)z_x = z$ is the required differential equation.

(5) $\phi(xy, z/x^2) = 0$

Let $u = xy$ & $v = z/x^2$

$\phi(u,v) = 0$

$\partial\phi/\partial u\,(y) + \partial\phi/\partial v\left(\frac{x^2 z_x - 2xz}{x^4}\right) = 0$ — (1)

$\partial\phi/\partial u\,(x) + \partial\phi/\partial v\left(\frac{x^2 z_x}{x^4}\right) = 0$ — (2)

$$\begin{pmatrix} y & \frac{x^2 z_x - 2xz}{x^4} \\ x & \frac{x^2 z_x}{x^4} \end{pmatrix}\begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} = 0$$

$\frac{x^2 y z_x}{x^4} - x^3 z_x + 2xz = 0$

$x^2 y z_x - x^7 z_x + 2x^5 z = 0$

$(x^7 - x^2 y)z_x = 2x^5 z$ is the required differential equation

~~(6) $z = (x+a)(y+b)$~~

(6) $z = ax + by + a^2 + b^2$

Differentiate w.r.t x & y

$\frac{\partial z}{\partial x} = a$ & $\frac{\partial z}{\partial y} = b$

substitute in the above equation

$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

is the required PDE.

(7) $z^2 = x^2/a^2 + y^2/b^2$

Differentiate w.r.t x & y

$2z z_x = 2x/a^2$ & $2z z_y = 2y/b^2$

$2z^2 z_x = 2/a^2$ & $2z^2 z_y = 2/b^2$

substitute in the above equation

$z^2 = x^2 z^2 z_x + y^2 z^2 z_y$ is the required PDE

(8) $\phi(xy + z^3, y+z) = 0$

Let $u = xy + z^3$ & $v = y + z$

$\partial\phi/\partial u\,(y + 3z^2 z_x) + \partial\phi/\partial v\,(z_x) = 0$

$\partial\phi/\partial u\,(x + 3z^2 z_y) + \partial\phi/\partial v\,(z_y) = 0$

$$\begin{pmatrix} y + 3z^2 z_x & z_x \\ x + 3z^2 z_y & z_y \end{pmatrix}\begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} = 0$$

$y z_y + 3z^2 z_x z_y + x z_x + 3z^2 z_x z_y$

$y z_y + x z_x + 6z^2 z_x z_y = 0$

is the required PDE

(8) $z = (x-a)^2 + (y-b)^2$

Differentiate w.r.t x & y

$z_x = 2(x-a)$ & $z_y = 2(y-b)$

substitute in the above equation

$$z = \cancel{\left(\frac{2x}{2}\right)^2 + \left(\frac{2y}{2}\right)^2}$$

(8) $z^2 = (x-a)^2 + (y-b)^2$

Differentiate w.r.t x & y

$2z z_x = 2(x-a)$ & $2z z_y = 2(y-b)$

$z^2 = (z z_x)^2 + (z z_y)^2$

$\therefore (z_x)^2 + (z_y)^2 = 1$

(7) $z^2 = x^2/a^2 + y^2/b^2$

$2z z_x = 2x/a^2$ & $2z z_y = 2y/b^2$

$\cancel{2} z z_x = x/a^2$ & $z z_y = y/b^2$

$z^2 = x^2\left(\frac{zx}{a^2}\right) + \quad z^2 = y^2\left(\frac{zx}{x}\right)$

(9) $z = f(2x-y) + g(3x+y)$ — (1)

Differentiate w.r.t x

$z_x = f'(2x-y)2 + g'(3x+y)(3)$

$\therefore z = 2f'(2x-y) + 3g'(3x+y)$

$z_{xx} = 4f''(2x-y) + 9g''(3x+y)$ — (2)

Differentiating w.r.t y

$z_y = f'(2x-y)(-1) + g'(3x+y)(1)$

$z_{yy} = f''(2x-y) + g''(3x+y)$ — (3)

Boundary value problem

1. $\partial u/\partial t = u + \frac{\partial u}{\partial x}$, given that $u = 4e^{-3x}$ when $t=0$

soln

Let $u(X,T) = X(x)T(t) = XT$

$XT' = XT + X'T$

$\frac{T'}{T} = 1 + \frac{X'}{X} = \alpha$

$\therefore T'/T = \alpha$ ——— (1)

$X'/X = \alpha - 1$ ——— (2)

From (1) $T'/T = \alpha$

$\ln T = \alpha t + c$

$\Rightarrow T(t) = e^{\alpha t + c} \Rightarrow K_1 e^{\alpha t}$

From (2) $X'/X = \alpha - 1$

$\ln x = \alpha x - x + D$

$X = e^{x(\alpha-1)+D} = K_2 e^{x(\alpha-1)}$

$TX = K_1 e^{\alpha t} \cdot K_2 e^{x(\alpha-1)}$

$= K e^{\alpha t + x(\alpha - 1)}$

$t = 0$

$= K e^{\alpha \cdot 0 + x(\alpha-1)}$

$= K e^{x(\alpha-1)} = 4e^{-3x}$

$\therefore K = 4 \quad \alpha = -2$

$TX = 4e^{-2t - 3x}$

(2) $\frac{\partial^2 u}{\partial x \partial y} = 4e^y \cos 2x$ given that at $y=0$, $\partial u/\partial x = \cos 2x$ and at $x = \pi$, $u = y^2$.

soln

$\partial u/\partial x(\partial/\partial y) = 4e^y \cos 2x$

$\partial u/\partial x = 4e^y \cos 2x + D(x)$

at $y=0$, $\partial u/\partial x = \cos 2x$

$\cos 2x = 4\cos 2x + D(x)$

$\therefore D(x) = -3\cos 2x$

$\partial u/\partial x = 4e^y \cos 2x - 3\cos 2x$

$u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + E(y)$

$y^2 = 2e^y \sin 2(\pi) - 3/2 \sin 2(\pi) + E(y)$

$\therefore E(y) = y^2$

$u(x,y) = 2e^y \sin 2x - 3/2 \sin 2x + y^2$

(3) $\partial^2 u/\partial x \partial t = \sin x \cos t$, if $\partial u/\partial x = 2x$ at $t = \pi/2$ and $u = 2\sin t$, at $x = \pi$

soln

$\partial u/\partial x(\partial/\partial t) = \sin x \cos t$

$\partial u/\partial x = \sin x \sin t + c(x)$

at $t = \pi/2$, $\partial u/\partial x = 2x$

$2x = \sin x \sin \pi/2 + c(x)$

$2x = \sin x + c(x)$

$\therefore c(x) = 2x - \sin x$

$\partial u/\partial x = \sin x \sin t + 2x + \sin x$

$u(x,t) = -\cos x \sin t + 2x + \cos x$

at $x = \pi$, $u = 2\sin t$

$u(\pi/2, t) = -\cos\pi/2 \sin t + 2\pi/2\, t + \cos\pi/2 + ...$

$= \sin t + \pi t - 1 = 2\sin t$

Q. $4\,\partial u/\partial x + \partial u/\partial y = 3u$, given that $u = 4e^{-3y}$ when $x = 0$

Soln

Let $u = X(x)T(y) = XY$

$4X'Y + XY' = 3(XY)$

$4\dfrac{X'}{X} + \dfrac{Y'}{Y} = 3$

$\dfrac{4X'}{X} = -\dfrac{Y'}{Y} + 3 = -\alpha$

$\dfrac{4X'}{X} = -\alpha \Rightarrow \ln X = -\alpha/4\, x + c$

$\Rightarrow X = e^{-\alpha/4 x + c} = K_1 e^{-\alpha/4 x}$ — (1)

$Y'/Y = \alpha + 3 \Rightarrow \ln Y = \alpha y + 3y$

$\Rightarrow Y = K_2 e^{(\alpha+3)y}$

$XY = K e^{-\alpha/4 x + (\alpha+3)y}$

at $x = 0$

$K e^{-\alpha/4(0) + (\alpha+3)y} = 4e^{-3y}$

$K = 4, \quad \alpha = -6$

$u = 4e^{3/2 x - 3y}$

(10) $z = f(x - at) + g(x + at)$

Differentiate w.r.t x

$z_x = f'(x - at) + g'(x + at)$

$z_{xx} = f''(x - at) + g''(x + at)$

Differentiate w.r.t y

$z_y = f'(x - at)(-a) + g'(x + at)\,a$

$z_{tt} = f''(x - at)\,a^2 + g''(x + at)\,a^2$

$z_{tt} = a^2[f''(x - at) + g''(x + at)]$

$z_{tt} = a^2 z_{xx}$ is the required PDE.

Q.

- $z^2 = x^2/a^2 + y^2/b^2$ —— ①

Differentiate w.r.t. x & y

$2z z_x = 2x/a^2 \Rightarrow 1/a^2 = z z_x/x$ — ②

$2z z_y = 2y/b^2 \rightarrow 1/b^2 = z z_y/y$ —— ③

Substitute ② & ③ in ①

$z^2 = x^2\left(z z_x/x\right) + y^2\left(z z_y/y\right)$

$z^2 = x z z_x + y z z_y$

$z = x z_x + y z_y$ is a required PDE.

- $\phi(xy + z^2, y+z) = 0$

Let $u = xy + z^2$ & $v = y+z$

$\phi(u, v)$

Differentiate w.r.t x

$\partial\phi/\partial u \cdot \partial u/\partial x + \partial\phi/\partial v \cdot \partial v/\partial x$

$\partial\phi/\partial u(y + 2z z_x) + \partial\phi/\partial v(z_x)$

Differentiate w.r.t y

$\partial\phi/\partial u \cdot \partial u/\partial y + \partial\phi/\partial v \cdot \partial v/\partial y$

$\partial\phi/\partial u(x + 2z z_y) + \partial\phi/\partial v(1 + z_y)$

$$\begin{vmatrix} y + 2z z_x & z_x \\ x + 2z z_y & 1 + z_y \end{vmatrix} \begin{pmatrix} \partial\phi/\partial u \\ \partial\phi/\partial v \end{pmatrix} = 0$$

$\Rightarrow (x - 2z) z_x - y z_y = y$ is a required PDE.

Q. $z(x,y) = e^x f(2y-3x)$ —— (1)

Differentiate w.r.t x

$z_x = e^x f'(2y-3x)(-3)$

~~Diff~~ $+ e^x f(2y-3x)$ —— (2)

Differentiate w.r.t y

$z_y = e^x f'(2y-3x)\,2$ —— (3)

$\Rightarrow f'(2y-3x) = \dfrac{z_y}{2e^x}$

$z_x = e^x \cdot \dfrac{z_y}{2e^x}(-3) + e^x f(2y-3x)$

$z_x = -\dfrac{3}{2} z_y + e^x f(2y-3x)$

From (1)

$f(2y-3x) = z/e^x$

$z_x = -3/2\, z_y + e^x \cdot z/e^x$

$z_x = -3/2\, z_y + z.$

Q. ~~Solve~~ Given that

$\partial^2 y/\partial t^2 = 4\,\partial^2 y/\partial x^2$; $y(x,0) = 0$,

$y(0,t) = y(\pi,t) = 0$, $\partial y/\partial t\,(x,0) = 3\sin x - 2\sin 5x$.

Now, let $Y = X(x)T(t) = XT$

$XT'' = 4X''T$

$\dfrac{T''}{4T} = \dfrac{X''}{X} = -\alpha^2$

$\dfrac{T''}{4T} = -\alpha^2$

$T'' + 4\alpha^2 T = 0$ —— (1)

$\dfrac{X''}{X} = -\alpha^2$

$X'' + \alpha^2 X = 0$ —— (2)

From (2) the auxiliary equa. is given by

$T(x) = A_1 \cos 2\alpha t + A_2 \sin 2\alpha t$

From (1) the auxilliary equa. are

$X(x) = B_1 \cos\alpha x + B_2 \sin\alpha x$

$y(x,t) = (B_1\cos\alpha x + B_2 \sin\alpha x)(A_1\cos 2\alpha t + A_2 \sin 2\alpha t)$

$y(0,t) = (B_1 \cos\alpha\cdot 0 + B_2 \sin\alpha\cdot 0)(A_1\cos 2\alpha t + A_2\sin 2\alpha t)$

$\Rightarrow B_1\cdot 1 + 0 = 0 \Rightarrow B_1 = 0$

$y(x,t) = B_2 \sin\alpha x(A_1\cos 2\alpha t + A_2 \sin 2\alpha t)$

$y(x,t) = \sin\alpha x(A\cos 2\alpha t + B\sin 2\alpha t)$

$y(x,0) = \sin\alpha x(A\cos 2\alpha\cdot 0 + B\sin 2\alpha\cdot 0)$

$\Rightarrow A\cdot 1 = 0 \Rightarrow A = 0$

$y(x,t) = \sin\alpha x(B\sin 2\alpha t)$

$= B\sin\alpha x \sin 2\alpha t$

$y(\pi,t) = B\sin\alpha\cdot\pi \sin 2\alpha t = 0$

$\Rightarrow \sin\alpha\pi = 0$

$\Rightarrow \alpha\pi = n\pi \Rightarrow \alpha = n$

$y(x,t) = B\sin nx \sin 2nt$

$\frac{d}{dt}(x,t) = B\sin nx \cdot 2n\cos 2nt$

$= 2nB\sin nx \cos 2nt$

$\frac{d}{dt}(x,0) = 2nB\sin nx \cos 2n\cdot 0$

$= 2nB\sin nx$

By the principle of superposition

$\frac{d}{dt}(x,0) = 2n_1B_1\sin n_1x + 2n_2B_2\sin n_2x$

$= 3\sin x - 2\sin 5x$

$\Rightarrow 2n_1B_1 = 3 \Rightarrow B_1 = 3/2$

$n_1 = 1$, $\cancel{B_2 = B_2} \Rightarrow B_2 =$

$\Rightarrow n_2 = 5$, $\cancel{2n_2}B_2 = -2$

$\Rightarrow 5B_2 = -1 \Rightarrow B_2 = -1/5$

Hence,

$\Rightarrow 2\cdot 1\cdot 3/2 \sin x - \cancel{2\cdot 5}\cdot 1/5 \sin 5x$

$\Rightarrow 3\sin x - 2\sin 5x$

$y(x,t) = B\sin nx \sin 2nt$

$= 3/2 \sin x \sin 2t - 1/5 \sin 5x$ - $\sin 10t$

Q Given that $\partial u/\partial t = 2\,\partial^2 u/\partial x^2$, $\partial u/\partial x(0,t) = 0$, $\partial u/\partial x(3,t) = 0$ and $\cancel{\partial u/\partial x}$ $u(x,0) = 3x$

Sol

Let $u = X(x)T(t) = XT$

$XT' = 2X''T$

$\frac{T'}{2T} = \frac{X''}{X} = -\alpha^2$

$T'/T = -2\alpha^2$ —— (1)

$X'' + \alpha^2 X = 0$ —— (2)

From (1) & (2), the auxiliary equation are given by

$\ln T = -2\alpha^2 t + c$

$T = e^{-2\alpha^2 t + c}$

$T = Ke^{-2\alpha^2 t}$

$X(x) = A_1\cos\alpha x + A_2\sin\alpha x$

$u(x,t) = Ke^{-2\alpha^2 t}(A_1\cos\alpha x + A_2\sin\alpha x)$

$= e^{-2\alpha^2 t}(A\cos\alpha x + B\sin\alpha x)$

$\partial u/\partial x = e^{-2\alpha^2 t}(\alpha B\cos\alpha x - \alpha A\sin\alpha x)$

$\partial u/\partial x(0,t) = e^{-2\alpha^2 t}(B\cos\alpha\cdot 0 - 0)$

$\Rightarrow B = 0$

$\partial u/\partial x(0,t) = Be^{-2\alpha^2 t}\alpha \cancel{\sin}\sin\alpha x$

$\partial u/\partial x(3,t) = Be^{-2\alpha^2 t}\alpha\sin\cancel{3\alpha}3x$

$\cancel{e^{-2\alpha^2 t} = e^0}$

$3\alpha = n\pi \Rightarrow \alpha = n\pi/3$